Remark on "What You See and What You Don't See:..."

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There is a general expression for the expected hidden tail moment that may be useful. It is this:

$$\mathbf{E}[\mu_{K_n,p}] = \frac{\mathbf{E}[K_{n+1}^p]}{n+1}. \tag{1}$$

(I'm assuming that the i.i.d X_k are strictly positive and have a continuous cdf.) Indeed, for a fixed $k > 0$ write

$$g(k) := \int_k^\infty x^p\, F(dx) = \mathbf{E}[X_{n+1}^p 1_{\{X_{n+1}>k\}}].$$

Then

$$\begin{aligned}
\mathbf{E}[\mu_{K_n,p}] &= \mathbf{E}\left[\int_0^\infty 1_{\{x>K_n\}} x^p\, F(dx)\right] \\
&= \mathbf{E}[g(K_n)] \\
&= \mathbf{E}\left[X_{n+1}^p 1_{\{X_{n+1}>K_n\}}\right] \\
&= \mathbf{E}\left[X_{n+1}^p 1_{\{X_{n+1}=K_{n+1}\}}\right]
\end{aligned}$$

By symmetry this last expectation is equal to

$$\mathbf{E}\left[X_k^p 1_{\{X_k=K_{n+1}\}}\right], \qquad k = 1, 2, \ldots, n.$$

Summing over $k \in \{1, 2, \ldots, n+1\}$ we obtain (1).

The expression (1) leads to a heuristic for the order of magnitude of $\mathbf{E}[\mu_{K_n,p}]$. If $\overline{F}(x) := 1 - F(x)$ is the tail for the X_k, then $V := \overline{F}(K_{n+1})$ has the same distribution as the minimum of a sample of $n+1$ uniform $(0, 1)$ random variables. As such $K_{n+1} = \overline{F}^{-1}(V)$ is roughly $\overline{F}^{-1}(1/(n+2))$, and so

$$\mathbf{E}[\mu_{K_n,p}] \approx \frac{\left[\overline{F}^{-1}(1/(n+2))\right]^p}{n+1},$$

for large n, which is consistent with your expression for $\mathbf{E}[\mu_{K_n,p}]$ in the power law case.